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Topological phases of quantum theories. Chern–Simons theory

M. Asorey

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain [1]

We analyze the vacuum structure (degeneracy, nodes and symmetries) of some quantum theories with special emphasis on the study of its dependence on the geometry and topology of the classical configuration space. The study of the topological limit shows that many low energy properties of those quantum theories can be inferred from the structure of their topological phases. After reviewing some simple pure quantum mechanical models (planar rotor, magnetic monopole and quantum Hall effect) we focus on the study of the rich relationship existing between topologically massive gauge theories and their topological phases, Chern–Simons theories. In particular we show that, although in a finite volume the degeneracy of the quantum vacuum of gauge theories depends on the topology of the underlying Riemann surface, in an infinite volume the vacuum is unique. Finally, the topological structure of Chern–Simons theory is analyzed in a covariant formalism within a geometric regularization scheme. We discuss in some detail the structure of the different metric dependent contributions to the Chern–Simons partition function and the associated topological invariants.

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1. Introduction

The relevant field configurations in the (euclidean) functional integral of a quantum field theory are rather singular. In general, they are distributions for space–time with dimension $D \geq 2$ [1]. However, the low energy properties are usually encoded by some classical smooth field configurations. For such a reason topological effects usually arise in the low energy regime. For instance, the structure of the quantum vacuum and the existence of spontaneous symmetry breaking depend very much on the geometry of the space of static classical configurations with minimal effective energy. Semiclassical solutions of the euclidean

[1] E-mail: asorey@cc.unizar.es, esteve@cc.unizar.es

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equations of motion also play a certain role in the analysis of the quantum tunneling between classical vacua and the existence of a mass gap. From a functional integral viewpoint the role of those smooth configurations can be understood in terms of the dominance of the weight of neighboring singular configurations with relative momentum bigger than an effective low momentum scale [2].

In general, the lagrangians of local field theories have two kinds of terms: ultralocal (without field derivatives) potential terms and generalized kinetic terms, which are responsible for the interaction and its propagation, respectively. Propagating terms are usually quadratic in fields and space–time derivatives. For those generic systems the classical properties which are relevant for the quantum theory are encoded by the minima of the potential term (static solutions, kinks, solitons, monopoles, etc.) or dynamic solutions which minimize the energy functional (instantons). Minima of the potential constitute the basic building blocks of the quantum vacuum, and instantons with non-trivial tunneling contribution are very relevant for the determination of its final structure.

A new type of physical phenomena arises when there is no ultralocal potential term and there are interactions linear in space–time derivatives [3–5]. In such a case the low energy regime has a richer structure and new topological effects arise. The dominant terms of the effective action at low energy are linear in space–time derivatives and therefore, if we neglect the irrelevant terms, the effective theory becomes singular from the canonical formalism viewpoint. The analysis of the constraints of the effective theory leads to a reduced phase space (moduli space) which can be quantized by means of geometric/holomorphic methods. The difference with the standard case (ultralocal potential + quadratic propagating term) is that in the later the low energy physics can be inferred from a semiclassical analysis of the effective action, which does not require an exact quantization, whereas in the former case, instead of averaging over collective coordinates it is necessary to quantize the reduced theory over its moduli space, although in many cases the semiclassical quantization is also exact. The infrared limit defines a new phase of the theory, the *topological phase*, which has a very peculiar (universal) behavior. Generically speaking, in this phase there are no local degrees of freedom and the only observables are topological invariants [6] (see ref. [7] for a review). Witten conjectured the consistency of quantum gravity in a topological phase [8]. The notion of topological phase is very different from the geometric phase, which generalizes the Aharonov–Bohm and Berry's phases [9] and should not be confused with it.

For such systems there is a dual relationship between the quantum field theory and its low energy effective theory which is very peculiar and merits some attention. Usually, the topological theory which emerges from the infrared limit (topological phase) is simpler and in many cases it is exactly solvable. Therefore it would be very interesting to learn how to extract information about the quantum theory from its topological phase. On the other hand, since topological theories are usually singular, fully fledged quantum theories can be used as ultraviolet regularizations of their topological limits. From this point of view it is also interesting to analyze the possible dependence (if any) of the topological observables defined in the topological theory on the method of ultraviolet regularization.

Although the low energy regime is dominated by smooth configurations the topological theory defined by the relevant terms of the effective action is more singular (e.g., dominant configurations in the functional integral are more singular distributions). After elimination of all spurious degrees of freedom and factorization of a divergent factor the functional integral becomes more regular, e.g., the Haussdorff dimension of the quantum fields is renormalized to zero. However, if the quantization is carried out before the elimination of the spurious degrees of freedom the corresponding quantum theory might exhibit an anomalous behavior induced by the quantum effects of non-topological degrees of freedom. In the case of Chern–Simons theory it has been shown that such an anomaly does not appear in the canonical formalism [10,11], but it is still unclear whether it arises or not in the covariant formalism by the fluctuation of non-physical degrees of freedom.

In this paper we analyze the low energy regime of some quantum systems with non-trivial topological phases in order to clarify the connection between topological theories and ordinary quantum theories defined on arbitrary manifolds. Special emphasis is put on the study of the dependence on the background metric of the manifold. The main purpose would be to understand the relationship between topologically massive Yang–Mills theories in $2+1$ dimensions and Chern–Simons theory defined on arbitrary three dimensional manifolds. We shall focus on the analysis of the partition function because it is the most sensitive observable under changes of space–time metrics.

The organization of the paper is as follows. In section 1 we analyze the low energy limit of some simple quantum systems with interactions linear in time derivatives. In particular, we point out the existence of a certain dependence on the regularization method. The same analysis carried out in section 3 for Yang–Mills theories with Chern–Simons interactions shows the uniqueness of the quantum vacuum in the infinite volume limit. Finally we discuss in section 4 the background metric dependence of the partition function of Chern–Simons theory in the covariant formalism using a geometric regularization of ultraviolet divergences which is related to topologically massive Yang–Mills theory. We show that the topological character is enhanced for some particular choice of the regulators and metrics of space–time manifolds.

2. Topological phases in quantum mechanics

Before discussing Chern–Simons theory we analyze some simpler similar pure quantum mechanical models. These systems have their own interest for different physical applications [12–15], although their role here is simply to illustrate and clarify the main features of topological phases of quantum theories.

The models describe the interaction of a point-like charged particle constrained to move in some submanifolds of $\mathbb{R}^3$ with a magnetic field.

The classical lagrangian is

$$L = \tfrac{1}{2}m\dot{\boldsymbol{x}}^2 + e\boldsymbol{A}\cdot\dot{\boldsymbol{x}}, \tag{2.1}$$

where m and e are the mass and charge of the particle and $\boldsymbol{A}$ is the vector potential of the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. The topological limit $m \to 0$ corresponds to the case where the lagrangian

$$L = e\boldsymbol{A}\cdot\dot{\boldsymbol{x}} \tag{2.2}$$

is linear in time derivatives [3–5].

The corresponding quantum system is defined by the following hamiltonian:

$$\mathbb{H} = -\frac{1}{2m}(\boldsymbol{\nabla} - \mathrm{i}e\boldsymbol{A})^2, \tag{2.3}$$

which in the topological limit $m \to 0$ becomes singular except in the subspace of null eigenvalues. Generically speaking, there are no null eigenvalues and the topological limit only makes sense after a (divergent) renormalization of $\mathbb{H}$ which cancels the ground state energy. The space of quantum states is then reduced to the subspace of ground states of $\mathbb{H}$ (*reduction of Hilbert space*).

When the particle is constrained to move in a submanifold of $\mathbb{R}^3$ we obtain a family of systems with different topological limits which depend on the geometry of the submanifold M and the form of the magnetic fields. In the general case the classical system is described on the phase space T^*M with the symplectic structure $\omega_0 + \omega$ defined by the sum of the canonical form ω_0 and the two-form ω of M associated to the magnetic field. Quantization is only possible if $(2\pi)^{-1}\omega$ is an integer form, i.e., $(2\pi)^{-1}[\omega] \in H^2(M,\mathbb{Z})$. In that case the quantum states are defined by sections of a line bundle $E(M,\mathbb{C})$ with a connection A whose curvature $\omega_A = dA = \pi^*\omega$ is the pull-back of ω by the projection map $\pi : E \to M$ of the bundle $E(M,\mathbb{C})$. If M is an oriented riemannian manifold the quantum hamiltonian is of the form

$$\mathbb{H} = -\frac{1}{2m}\Delta_A, \tag{2.4}$$

where $\Delta_A = d_A^* d_A$, $d_A^* = (-1) * d_A *$ is the adjoint of the covariant differential d_A and $*$ is the Hodge operator associated to the riemannian structure g of M.

2.1. PLANAR ROTOR

The first model describes the interaction of a charged particle moving on a circle $S^1 = \{x \in \mathbb{R}^2; ||x|| = 1\}$ with a magnetic field vanishing on S^1 but with non-trivial magnetic flux across S^1 [12]. The corresponding vector potential can be expressed in cylindric coordinates by $A_\theta = \phi/2\pi$, $A_r = 0, A_z = 0$, and satisfies the two conditions: $\boldsymbol{B} = \nabla \times \boldsymbol{A} = 0$ for $r \neq 0$ and

$$\int_{S^1} \boldsymbol{A}\, d\gamma = \int_D \boldsymbol{B}\, d\boldsymbol{\sigma} = \phi \neq 0.$$

The quantum dynamics is given by the hamiltonian

$$\mathbb{H} = -\frac{1}{2m}\left(\partial_\theta - \mathrm{i}\frac{e\phi}{2\pi}\right)^2 \tag{2.5}$$

with periodic boundary conditions $\psi(0) = \psi(2\pi)$. The spectrum of $\mathbb{H}$ is

$$E_n = \frac{1}{2m}(n - \varepsilon)^2, \tag{2.6}$$

where $\varepsilon = e\phi/2\pi$.

The ground state is unique for $\varepsilon \neq (n + \frac{1}{2})$. However, for $\varepsilon = (n + \frac{1}{2})$ the system exhibits a very peculiar behavior [12], which is reflected by the fact that the ground state is doubly degenerate. The existence of this degeneracy can be explained by the presence of complex interactions which prevent the application of the min–max principle to prove the uniqueness of the ground state. We remark that the classical SO(2) symmetry is preserved after quantization. The ground state subspace spans two irreducible representations of SO(2) for $\varepsilon = (n + \frac{1}{2})$ and only one in the other cases.

Consequently, in the topological limit $m \to 0$ the infinite dimensional Hilbert space is reduced to a finite dimensional one

$$\mathcal{H}^0_\varepsilon = \begin{cases} \mathbb{C}, & \varepsilon \neq (n + \frac{1}{2}), \\ \mathbb{C}^2, & \varepsilon = (n + \frac{1}{2}). \end{cases} \tag{2.7}$$

From a classical point of view the reduced phase space is a single point in both cases because $\omega = 0$. Therefore, quantization from this reduced phase space leads to the same Hilbert space $\mathcal{H}^0_\varepsilon = \mathbb{C}$ and hamiltonian $\mathbb{H} = 0$ in both cases. Thus, the peculiar behavior of the case $\varepsilon = (n + \frac{1}{2})$ is not observed if the constraints are eliminated before quantization.

The topological phase of the quantum system is independent of the metric of the world (time) line because of the time reparametrization invariance, and is also independent of the metric of the target space S^1 used in the regularization. In this case both classical properties are preserved under quantization unlike for the systems to be considered below. The difference between the two types

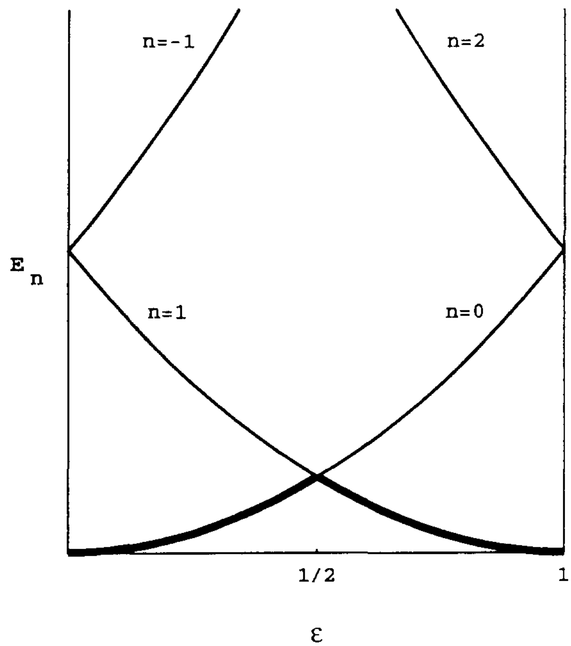

Fig. 1. Low energy levels of the planar rotor hamiltonian for $0 \leq \varepsilon \leq 1$. The thick curves represent the ground state energies and the cusp at $\varepsilon = 1/2$ the transition point where the ground state is degenerate.

of topological quantum phases generated by the planar rotor is encoded by a (metric independent) topological invariant

$$\int_{S^1} A\, d\gamma .$$

2.2. MAGNETIC MONOPOLE

We now consider the interaction of a charged particle constrained to move on a two dimensional sphere $S^2 = \{\boldsymbol{x} \in \mathbb{R}^3; ||\boldsymbol{x}|| = a\}$ with a Dirac magnetic monopole

$$\boldsymbol{B} = g\boldsymbol{x}/||\boldsymbol{x}||^3. \tag{2.8}$$

Although there is no global regular vector potential A over the whole sphere S^2 with $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, there are vector potentials of B,

$$A_j^{\pm} = \pm g \frac{\varepsilon_{j3k} x^k}{||x||(||x|| \pm x^3)},$$

which are only singular at the north and south poles, respectively.

The quantum hamiltonians

$$\mathbb{H} = -\frac{1}{2m}(\nabla - ieA^{\pm})^2 \tag{2.9}$$

are not univocally defined as self-adjoint operators on a functional Hilbert space unless the magnetic flux across the sphere S^2,

$$e\int_{S^2} \boldsymbol{B}\, d\boldsymbol{\sigma} = 4\pi e g\,,$$

is quantized, $k = 2eg \in \mathbb{Z}$ (Dirac's quantization condition [13]). This topological (metric independent) condition implies that $A^{\pm}$ defines a connection on a line bundle $E(S^2,\mathbb{C})$ with first Chern characteristic number $c_1(E) = k$ [16].

Since the metric of S^2 induced from $\mathbb{R}^3$ is SO(3) invariant there is a horizontal lift of this symmetry to an action of its universal covering group SU(2) on $E(S^2,\mathbb{C})$. Because of the non-trivial magnetic flux the generators of this action pick up an additional term

$$L_s = \varepsilon_{ijs} x^i \nabla_A^j - \tfrac{1}{2} k x^s / \|\boldsymbol{x}\|\,, \tag{2.10}$$

which is necessary to preserve the SO(3) symmetry of the quantum hamiltonian $[\mathbb{H}, L_r] = 0$ and the SO(3) Lie algebra commutation relations

$$[L_r, L_s] = \varepsilon_{rst} L_t. \tag{2.11}$$

The induced representation of SU(2) on the space of sections of $E(S^2,\mathbb{C})$ can be decomposed into irreducible representations $\mathcal{H}_k^l$ parametrized by the eigenvalues of the second Casimir operator $L^2 = L_i L^i$,

$$\tfrac{1}{4}(|k| + 2l)(|k| + 2l + 2)\,, \quad l \in \mathbb{N}\,,$$

whose degeneracy is given by the Frobenius reciprocity theorem,

$$\dim \mathcal{H}_k^l = |k| + 2l + 1. \tag{2.12}$$

The spectrum of the hamiltonian can be easily obtained from symmetry arguments or by the following explicit calculation.

In the complex coordinates

$$z = a e^{i\varphi} \tan\theta/2\,, \qquad \bar{z} = a e^{-i\varphi} \tan\theta/2\,,$$

the hamiltonian (2.9) reads

$$\mathbb{H} = -\frac{1}{2m}\left[\left(1 + \frac{z\bar{z}}{a^2}\right)^2 \partial\bar{\partial} + \frac{k}{2a^2}\left(1 + \frac{z\bar{z}}{a^2}\right)(z\partial - \bar{z}\bar{\partial}) - \frac{k^2}{4a^4} z\bar{z}\right]. \tag{2.13}$$

The spectrum can be easily obtained by means of the following similarity transformation :

$$\psi(z,\bar{z}) = (1 + z\bar{z}/a^2)^{-k/2} \xi(z,\bar{z})\,, \tag{2.14}$$

which transforms the hamiltonian (2.13) into

$$\mathbb{H}' = -\frac{1}{2m}\left[\left(1+\frac{z\bar{z}}{a^2}\right)^2\partial\bar{\partial} - \frac{k}{a^2}\left(1+\frac{z\bar{z}}{a^2}\right)z\bar{\partial} - \frac{k}{2a^2}\right]. \tag{2.15}$$

The eigenfunctions of $\mathbb{H}'$ are

$$\xi_n^l(z,\bar{z}) = z^j P_l^{(j,|k|-j)}\left(\frac{a^2 - z\bar{z}}{a^2 + z\bar{z}}\right), \quad \begin{array}{l} l = 0,1,2,\dots, \\ j = -l,-l+1,\dots,l+|k|, \end{array} \tag{2.16}$$

where $P_l^{(i,j)}$ $(i,j \geq -l)$ are the Jacobi polynomials [17]. The corresponding eigenvalues

$$E_l = \frac{1}{2ma^2}\left[|k|(l+\tfrac{1}{2}) + l(l+1)\right], \quad l = 0,1,2,\dots,$$

have degeneracies

$$2l + |k| + 1,$$

which are in agreement with the Frobenius reciprocity theorem requirements (2.12). In particular, the ground states are expanded by the analytic functions

$$\xi(z) = z^l, \quad l = 0,\dots,|k|,$$

and have non-trivial degeneracy, $|k| + 1$, whenever the magnetic monopole charge is non-null. Since the bundle $E(\mathrm{S}^2,\mathbb{C})$ is non-trivial for $k \neq 0$, every stationary state must have nodes, i.e., vanishing points. In the holomorphic representation the origin $z = 0$ and infinity $z = \infty$ are nodes of the basis of eigensections (2.16) of $\mathbb{H}$.

In the topological limit $m \to 0$ the Hilbert space is reduced to the finite dimensional space

$$\mathcal{H}_k^0 = \{\xi(z) = \textstyle\sum_{l=0}^{|k|} c_l z^l\}$$

of analytic sections of $E(\mathrm{S}^2,\mathbb{C})$ which are holomorphic with respect to the complex structure induced by the connection A, the magnetic field B and the riemannian metric of S^2. The Hilbert space $\mathcal{H}_k^0$ spans an irreducible representation of SU(2) with angular momentum $|k|/2$ and dimension $\dim \mathcal{H}_k^0 = |k| + 1$. The result of the topological limit is in agreement with that obtained by holomorphic quantization [18] from the reduced phase space $(\mathrm{S}^2, \omega = 2\pi i k\, d\bar{z} \wedge dz)$. However, this result might depend on the choice of the metric of the configuration (target) space. In this example the background metric of S^2 is maximally symmetric (SO(3)-invariant), but for a generic metric the degeneracy of the ground state is lower. This fact shows that a good prescription for using the topological limit as a method of quantization of a topological theory should be based on the most symmetrical metric of the configuration space in order to preserve as much classical symmetries as possible. Even in such a case some quantum anomalies might appear in the space of quantum states, which require a central extension of the classical symmetry group. In order to obtain a (target space) metric independent quantum theory not only a renormalization of the vacuum energy is required, but also a metric dependent operatorial renormalization of the hamiltonian, which is not usually considered in standard renormalization scheme prescriptions.

2.3. HALL EFFECT ON A TORUS

We now consider the interaction of a charged particle moving on a torus T^2 with a constant magnetic field, $\|\boldsymbol{B}\| = B = cte$. In complex coordinates $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$ the quantum hamiltonian

$$\mathbb{H} = -\frac{1}{2m}\left[4\partial\bar{\partial} + eB(z\partial - \bar{z}\bar{\partial}) - \tfrac{1}{4}e^2B^2z\bar{z}\right] \tag{2.17}$$

is uniquely defined as a self-adjoint operator iff the magnetic flux across the torus

$$\int_{T^2} \boldsymbol{B}\, d\sigma = B$$

is quantized, $k = eB/2\pi \in \mathbb{Z}$ (Dirac's condition). This topological constraint implies that the vector potential $A_i = -B\, \varepsilon_{ij}x^i/2$ defines a connection on a line bundle $E(T^2, \mathbb{C})$ with first Chern number $c_1(E) = k$. The quantum states are the L^2-sections of $E(T^2, \mathbb{C})$. Continuous sections of E satisfy the boundary conditions [#1]

$$\begin{aligned} \psi(z+1, \bar{z}+1) &= e^{k\pi(z-\bar{z})/2}\psi(z,\bar{z}), \\ \psi(z+i, \bar{z}-i) &= e^{-ik\pi(z+\bar{z})/2}\psi(z,\bar{z}), \end{aligned} \tag{2.18}$$

and because of the non-triviality of E (for $k=0$) must have nodes at some points of T^2 for non-vanishing magnetic flux. The classical translation $U(1)^2$ symmetry of the planar metric of T^2 becomes anomalous when lifted to the line bundle E. The minimal lift of the corresponding generators does not commute with the quantum hamiltonian (2.17) and does not satisfy the translation Lie algebra commutation relations. A central extension of the Lie algebra $U(1)^2$ which formally commutes with the quantum hamiltonian (2.17) can be obtained by adding a new term to the infinitesimal generators

$$p = \partial + \tfrac{1}{2}k\pi\bar{z}, \qquad \bar{p} = \bar{\partial} - \tfrac{1}{2}k\pi z, \tag{2.19}$$

in a similar way to the SO(3) symmetry in the case of a magnetic monopole (2.10). The corresponding quantum Lie algebra is a Heisenberg algebra,

$$[p, \bar{p}] = -k\pi. \tag{2.20}$$

[#1] There is a phase ambiguity in the choice of boundary conditions due to the non-simply connected character of the torus (Aharonov–Bohm phase). The prescription (2.18) (trivial phases) corresponds to the standard notation in the description of the quantum Hall effect [15] and has the nice property of preserving general covariance.

The generators of translation symmetries (2.19) cannot be truly considered as quantum symmetries of the hamiltonian $\mathbb{H}'$ because, although they formally commute with $\mathbb{H}'$, they do not preserve the boundary conditions (2.18). In this way translation invariance is anomalous in the sense advocated by Manton and Esteve. The existence of this anomaly explains why the energy levels of $\mathbb{H}'$ can be finitely degenerate whereas non-trivial irreducible representations of the Heisenberg algebra (2.20) are infinite dimensional. In the case of the infinite plane the anomaly does not exist, which implies an infinite degeneracy of the energy levels. The spectrum of the hamiltonian (2.17) can be obtained by means of a similarity transformation

$$\psi(z, \bar{z}) = e^{k\pi z\bar{z}/2}\xi(z, \bar{z}), \tag{2.21}$$

which yields

$$\mathbb{H}' = -\frac{1}{2m}\left[4(\partial - k\pi\bar{z})\bar{\partial} - 2k\pi\right], \tag{2.22}$$

with

$$\begin{aligned} \xi(z+1, \bar{z}+1) &= e^{i\pi kz + k\pi/2}\xi(z, \bar{z}), \\ \xi(z+i, \bar{z}-i) &= e^{-i\pi kz + k\pi/2}\xi(z, \bar{z}) \end{aligned} \tag{2.23}$$

as boundary conditions. The eigenfunctions can be expressed as infinite sums of eigenfunctions of a harmonic oscillator which satisfy the boundary conditions (2.23). Their eigenvalues are given by Landau levels

$$E_n^j = \frac{2\pi k}{m}(n + 1/2), \quad \begin{array}{l} n = 0, 1, 2, \ldots, \\ j = -n, -n+1, \ldots, |k| - 1, \end{array} \tag{2.24}$$

with degeneracy $n + |k|$. The ground state degeneracy $|k|$ can be understood by the failure of the min–max principle in the presence of complex interactions. The corresponding eigenfunctions are the holomorphic sections of $E(\mathrm{T}^2, \mathbb{C})$ (i.e. theta functions) with respect to the complex structure induced by the vector potential A, and the complex structure of the torus T^2,

$$\begin{aligned} \xi_0^j(z) &= e^{k\pi z^2/2}\Theta\begin{bmatrix} j/|k| \\ 0 \end{bmatrix}(|k|z, i|k|) \\ &= e^{k\pi z^2/2}\sum_{l\in\mathbb{Z}+j/|k|} e^{-\pi|k|l^2 + 2\pi i|k|lz}, \\ & j = 0, 1, 2, \ldots, |k| - 1. \end{aligned}$$

In the topological limit the Hilbert space is reduced to the ground state subspace

$$\mathcal{H}_k^0 = \{\xi(z); \textstyle\sum_{j=1}^{|k|} c_j \Theta_{|k|}^j(z)\}$$

if the vacuum energy is renormalized to zero. The results of this quantization method coincides with that obtained via holomorphic quantization from the reduced phase space.

In the present approach the result is again independent of the world line metric but it does depend on the (target space) T^2 metric. In fact, the reduced Hilbert space is smaller when the T^2 metric is not translation invariant. Once more this symmetry (although anomalous) is the guiding principle to quantize in agreement with holomorphic quantization. Actually, the same result is obtained for any other flat metric of T^2 corresponding to a different point τ in moduli space. In this case the vacuum states are also given in terms of the corresponding theta functions [19],

$$\begin{aligned} \xi_\tau^j(z) &= e^{k\pi z^2/2\,\mathrm{Im}\,\tau}\Theta\begin{bmatrix} j/|k| \\ 0 \end{bmatrix}(|k|z,|k|\tau) \\ &= e^{k\pi z^2/2\,\mathrm{Im}\,\tau}\sum_{l\in\mathbb{Z}+j/|k|} e^{i\tau\pi|k|l^2+2\pi i|k|lz}\,, \\ j &= 0,1,2,\ldots,|k|-1\,, \end{aligned}$$

and

$$\dim\mathcal{H}_k^\tau = \dim\mathcal{H}_k^0 = |k|\,.$$

Absolute independence of the T^2 metric would require a non-standard renormalization of the regularized hamiltonian modifying its operator structure.

Let us analyze, for completeness, the holomorphic quantization approach [18]. A classical constraint analysis of singular systems described by lagrangians of the form (2.1) shows that the conditions

$$p_i = eA_i\,, \quad i = 1,2\,, \tag{2.25}$$

are second class constraints,

$$\{p_i - eA_i, p_j - eA_j\}_{\mathrm{PB}} = eB\varepsilon_{ij} \neq 0\,, \quad i = 1,2\,. \tag{2.26}$$

If we perform a non-canonical transformation of the phase space $T^*(T^2)$,

$$x'^i = x^i\,, \qquad p_i' = p_i - eA_i\,, \quad i = 1,2\,, \tag{2.27}$$

the symplectic structure of $T^*(T^2)$ becomes

$$\omega' = \sum_{i=1,2} dp_i' \wedge dx^i + eB\,dx^1 \wedge dx^2. \tag{2.28}$$

The constraints in the new coordinates (x^i, p_i) read

$$p_i' = 0\,, \quad i = 1,2\,. \tag{2.29}$$

The classical method of dealing with second class constraints proceeds by restriction to the constraints submanifold $p_i' = 0$, $i = 1,2$. In this case it can be identified with the configuration space T^2 endowed with the symplectic structure defined by the magnetic field $F = 2\pi k\,dx^1 \wedge dx^2$, which is the restriction of the symplectic form ω' to the constraints manifold T^2. There are not further constraints.

Holomorphic quantization of the reduced phase space T^2 gives rise to a finite dimensional Hilbert space

$$\mathcal{H}_k^0 = \{\xi : T^2 \to E; \xi \text{ is analytic}\},$$

whose quantum states ξ are the holomorphic sections of a line bundle $E(T^2, \mathbb{C})$ with Chern class number $c_1(E) = k$.

The dimension of the space of quantum states is given by the Riemann–Roch formula

$$\dim \mathcal{H}_k^0 = \frac{1}{8\pi} \int_{T^2} \sqrt{g} R + \frac{1}{2\pi} \int_{T^2} F = k, \tag{2.30}$$

and the quantum hamiltonian is trivial, $H = 0$, as corresponds to a topological theory. The same Riemann–Roch formula gives account of the right degeneracy $(k+1)$ of the vacuum states in the monopole case of section 2.2. There is a similar formula for magnetic holomorphic fields on higher genus (g) Riemann surfaces[#2] [20],

$$\dim \mathcal{H}_k^0 = \frac{1}{8\pi} \int_{T^2} \sqrt{g} R + \frac{1}{2\pi} \int_{T^2} F + \dim \mathcal{H}^0_{2g-2-|k|}. \tag{2.31}$$

In the topological limit of the quantum system associated to the Hall effect we have obtained the same results in a different way. One interesting aspect of the geometrical method based on the topological limit is that it shows the way physical constraints arise in the topological phase. Before taking the topological limit the momentum operators are given by

$$\mathrm{i}\Pi = \partial - \tfrac{1}{2}\pi k \bar{z}, \qquad \mathrm{i}\overline{\Pi} = -\bar{\partial} - \tfrac{1}{2}\pi k z, \tag{2.32}$$

or

$$\mathrm{i}\Pi' = \partial - \pi k \bar{z}, \qquad \mathrm{i}\overline{\Pi}' = -\bar{\partial}, \tag{2.33}$$

once the similarity transformation (2.21) has been carried out. In the subspace of ground states the first constraint $\overline{\Pi}' = 0$ is satisfied in the strong operatorial sense. However, the second constraint $\Pi' = 0$ is only satisfied in a weak sense as the Lorentz condition in QED in the Gupta–Bleuler formalism: its expectation value vanishes,

$$(\xi_1, \Pi \xi_2) = \int_{T^2} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi i} \mathrm{e}^{-\pi k z \bar{z}} \bar{\xi}_1(\bar{z})(\partial - \pi k \bar{z}) \xi_2(z) = 0, \tag{2.34}$$

on ground states. Therefore in the topological limit, since $\mathcal{H} \to \mathcal{H}^0_{|k|}$, both constraints are satisfied in the strong operatorial sense. Notice that the second constraint $\Pi' = 0$ also arises in holomorphic quantization from the commutation

[#2] The last term in (2.31) vanishes for $2g - 2 - |k| \leq 0$ (Kodaira's vanishing theorem) but not in general, as erroneously indicated in ref. [21].

relations (2.20) implied by the non-trivial symplectic structure F of T^2. However, in the geometrical approach different topological limits impose further constraints derived from a different operator ordering prescription or a different choice of (target space) riemannian metric.

2.4. QUANTUM HALL EFFECT ON THE INFINITE PLANE

We now consider the same system with the charged particle moving on the plane $\mathbb{R}^2$ under the effect of a constant magnetic field. The quantization of the system proceeds in a similar manner to the previous case, but without the requirement of quantization of the magnetic flux across the plane. The spectrum of the hamiltonian (2.22) is given by the classical Landau levels

$$E_n^j = \frac{2\pi k}{m}(n + 1/2), \quad \begin{array}{l} n = 0, 1, 2, \dots, \\ j = -n, -n+1, \dots, \end{array} \tag{2.35}$$

which now are infinitely degenerate due to the absence of boundary conditions. The eigenfunctions can be expressed in this case in terms of generalized Laguerre polynomials [22],

$$\xi_n^j(z, \bar{z}) = z^j L_n^j(2\pi k|z|^2). \tag{2.36}$$

The above spectrum can be obtained from that of the torus in the infinite volume limit when the magnetic field $B \to \infty$ in such a way that the density of magnetic flux is held constant. In the same way, it can also be obtained from the monopole case when the radius of the sphere a and the charge of the magnetic monopole k go to infinity keeping the ratio k/a^2 constant (see, e.g., ref. [17]). In both cases the limit is well defined and the eigensections of the sequences of line bundles $E(T^2, \mathbb{C})$ and $E(S^2, \mathbb{C})$ tend to square integrable functions of $L^2(\mathbb{R}^2)$. The degeneracy of all energy levels grows with the Chern number k of E towards the infinite Landau degeneracy. This phenomenon does not occur in Chern–Simons theory, as will be shown in the next section, where the infinite volume limit of the theory on a torus reduces the subspace of ground states to the single unique ground state over $\mathbb{R}^2$. In that case the degeneracy disappears in the infinite volume limit. The analogous phenomenon here arises in the limit of infinite volume for the torus when the magnetic charge is kept constant.

The topological phase ($m \to 0$) corresponds to the system described by the low lying Landau levels,

$$\mathcal{H}_k^0 = \{\xi : \mathbb{R}^2 \to \mathbb{C};\ \bar{\partial}\xi = 0,\ \text{i.e., } \xi \text{ is analytic}\}.$$

The main peculiarity of this phase is that it has an infinite dimensional Hilbert space, which does not describe local interactions. The space of quantum states has a natural (ground) ring structure and describes an infinite dimensional representation of the Heisenberg group which emerges as the quantum symmetry associated with the classical translation invariance. Finally, it is easy to see that the structure of the topological phase changes when we consider a different riemannian metric in $\mathbb{R}^2$. Generically, the Heisenberg group does not leave the space of quantum states invariant.

3. Topological phases of (2 + 1)-dimensional gauge theories. Chern–Simons theory

In $2+1$ space–time dimensions gauge theories have a very interesting infrared structure. The presence of a Chern–Simons gluonic interaction, which can arise as an effective interaction induced by massless fermions, implies the existence of gauge invariant phases with massive gluons [23,25]. In such a case, although the gauge interaction is short-ranged, the quantum vacuum exhibits a very rich topologically dependent structure which reveals the existence of non-trivial topological phases [24].

In this section we shall review some aspects of the vacuum structure of topologically massive Yang–Mills theory (degeneracy, vacuum nodes, anomalous angular momentum) and its topological phase (Chern–Simons theory). The approach will be analogous to the one used in section 2 for topological quantum mechanics, but with some interesting physically meaningful differences. Since pure Chern–Simons theory is exactly solvable for compact Lie groups [10], its interpretation as a topological limit of a massive Yang–Mills theory provides very interesting information about the structure of the vacuum of the massive theory. On the other hand the massive theory can be considered as a regularization of the pure Chern–Simons theory [11,25–27], which might be useful for the analysis of singular observables and possible ambiguities of the theory.

The massive Yang–Mills action

$$S_A(A) = \frac{ik}{4\pi}\int_{M_3} \mathrm{tr}(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A) + \frac{k}{8\pi A}\|F(A)\|^2, \tag{3.1}$$

is not univocally defined for gauge fields A of any principal bundle $P(M, \mathrm{SU}(N))$. The value of the Chern–Simons term depends on the section of P chosen in the expression (3.1) and is not invariant under large gauge transformations of that section. The variation of $S_A(A)$ is $2\pi k$ times the winding number of the gauge transformation. Therefore, a consistent definition of the euclidean functional integral is only possible for integer values of the Chern–Simons charge k. We use the compact notation of ref. [28], where $\|\;\|$ denotes the norm associated to the scalar products of p-forms,

$$(\tau, \eta) = -2\int_{M_3} \mathrm{tr}\, \tau\wedge *\eta, \tag{3.2}$$

and $*$ is the Hodge operator associated to the (oriented) space–time metric (M_3, g).

3.1. CANONICAL QUANTIZATION

In the case of space–times of the form $M_3 = \Sigma \times \mathbb{R}$ with a direct product metric it is possible to develop a canonical approach. If we consider the temporal gauge ($A_0 = 0$), the only degrees of freedom are the spatial components A of the gauge fields $A = (A, A_0)$ with their momenta constrained in the phase space $T^*(\mathcal{A}_\Sigma)$ by the Gauss law

$$d_A^*(\Pi + \tfrac{1}{2}\kappa^{-2} * A) = \kappa^{-2} * F(A). \tag{3.3}$$

Here $\mathcal{A}_\Sigma$ denotes the configuration space of two dimensional gauge fields A on the Riemann surface Σ with $\mathrm{SU}(N)$ as structure group, κ is the Chern–Simons coupling constant $\kappa = \sqrt{4\pi/k}$ and $d_A^* = - * d_A *$ is the adjoint of the covariant derivative d_A operator with respect to the scalar product defined on $\mathcal{A}_\Sigma$ by (3.2).

The classical hamiltonian is given by

$$H = \tfrac{1}{2}\kappa^2 \Lambda \left\| \Pi + \frac{1}{2\kappa^2} * A \right\|^2 + \frac{1}{2\kappa^2 \Lambda} \|F(A)\|^2. \tag{3.4}$$

In the Schrödinger representation canonical quantization gives the following prescription for the momentum operator:

$$\mathbb{P} = -\mathrm{i}\,\delta/\delta A. \tag{3.5}$$

The quantum hamiltonian $\mathbb{H}$ is then obtained by introducing such a prescription for the quantum momentum $\mathbb{P}$ into the expression (3.4) of the classical hamiltonian. There is no ordering problem in the kinetic term because all orderings give rise to the same quantum operator $\mathbb{H}$. The quantum states are given the complex functionals $\psi(A)$ on $\mathcal{A}_\Sigma$ (from here on denoted $\mathcal{A}$, for simplicity), which satisfy the quantum Gauss law condition

$$-\mathrm{i} d_A^* \frac{\delta}{\delta A} \psi(A) = \frac{1}{2\kappa^2} * dA\, \psi(A). \tag{3.6}$$

This condition has a simple geometric interpretation in terms of the hermitian $\mathrm{U}(1)$ connection defined by

$$\widetilde{\alpha}_\kappa = \frac{1}{2\kappa^2} * A + \frac{1}{\kappa^2} d_A G_A * F(A), \tag{3.7}$$

with $G_A = (d_A^* d_A)^{-1}$. Actually, the quantum Gauss law condition (3.6) can be written as

$$d_A^* \nabla_{\widetilde{\alpha}_\kappa} \psi(A) = 0 \tag{3.8}$$

with

$$\nabla_{\widetilde{\alpha}_\kappa} = \delta/\delta A + \mathrm{i}\widetilde{\alpha}_\kappa ,$$

which means that the quantum states are covariantly constant along the gauge fibers with respect to the connection $\widetilde{\alpha}_\kappa$. The existence of non-trivial solutions of the quantum Gauss condition (3.8) is possible iff the connection $\widetilde{\alpha}_\kappa$ is trivial along the orbits of the group of gauge transformations $\mathcal{G}$. The curvature two-form of $\widetilde{\alpha}_\kappa$,

$$\widetilde{\Omega}_\kappa(\widetilde{\tau},\widetilde{\eta}) = -\frac{1}{2\kappa^2}(\widetilde{\tau},*\widetilde{\eta}) + \frac{1}{\kappa^2}(G_A*[\widetilde{\tau},\widetilde{\eta}],*F(A)), \tag{3.9}$$

vanishes for vectors $\widetilde{\tau},\widetilde{\eta} \in T_A(\mathcal{A})$ tangent to the gauge fibers $\widetilde{\tau} = d_A\phi$. However, $\widetilde{\alpha}_\kappa$ is trivial only if the holonomy group associated to any closed curve contained in a gauge orbit is trivial. This is only possible if the projection Ω_κ of the curvature two-form $\widetilde{\Omega}_\kappa$ to the space of gauge orbits $\mathcal{M} = \mathcal{A}/\mathcal{G}$,

$$\Omega_\kappa(\tau,\eta) \mathrel{\dot{=}} \widetilde{\Omega}_\kappa(\widetilde{\tau}^{\mathrm{h}},\widetilde{\eta}^{\mathrm{h}}), \tag{3.10}$$

belongs (modulo a factor 2π) to an integer cohomology class of $\mathcal{M}$, i.e.,

$$(2\pi)^{-1}[\Omega] \in H^2(\mathcal{M},\mathbb{Z}). \tag{3.11}$$

$\widetilde{\tau}^{\mathrm{h}}$ denotes the horizontal component of any tangent vector $\widetilde{\tau} \in T_A(\mathcal{A})$ with projection $\tau \in T_{[A]}(\mathcal{M})$ which is orthogonal to the gauge fiber at A, i.e., $\widetilde{\tau}^{\mathrm{h}} = P_A\widetilde{\tau}$, $P_A = (I - d_A G_A d_A^*)$ being the corresponding orthogonal projector.

Condition (3.11) is satisfied if and only if the Chern–Simons charge k is an integer. In this way the quantization condition of k also arises in the canonical formalism. Although the above derivation of this consistency condition is very different from that given in the covariant functional integral formalism, both have a common origin: gauge invariance under large gauge transformations [29]. Because of the triviality of $\widetilde{\alpha}_\kappa$ when k is an integer, the action of the group of gauge transformations $\mathcal{G}$ can be globally lifted to an action on the line bundle $\mathcal{A}\times\mathbb{C}(\mathcal{A},\mathbb{C})$. The Gauss law implies the invariance of the quantum states under this action and then the quantum states can be completely characterized by sections of the line bundle $\mathcal{E}_k(\mathcal{M},\mathcal{G})$ defined by the gauge orbits $\mathcal{E}_k = \mathcal{A}\times\mathbb{C}/\mathcal{G}$ of such an action. In the same way, the connection $\widetilde{\alpha}_\kappa$ of $\mathcal{A}\times\mathbb{C}$ projects down to a connection α_κ in $\mathcal{E}_k$ and the quantum hamiltonian can be expressed as an operator

$$\mathbb{H} = \tfrac{1}{2}\kappa^2 A\|\nabla_{\alpha_\kappa}\|^2 + \frac{1}{2\kappa^2 A}\|F(A)\|^2 + \frac{A}{2\kappa^2}(G_A*F(A),*F(A)) \tag{3.12}$$

acting on the sections of $\mathcal{E}_k$. The Chern class of the line bundle $\mathcal{E}_k(\mathcal{M},\mathcal{C})$ is non-trivial, $c_1(\mathcal{E}_k) = k$. Therefore the quantum dynamics of the topologically massive Yang–Mills theory is very similar to that of the magnetic monopole on S^2 and the quantum Hall effect on a torus. There are however some differences due the presence of additional interacting terms, and the infinite dimensional character of the Yang–Mills configuration space $\mathcal{M}$, which generates some ultraviolet divergences and require renormalization. The similarity with the quantum mechanical models suggests the existence of low energy physical effects related to the topological structure of the orbit space. We summarize in the next section the main relevant topological properties of the orbit space $\mathcal{M}$ of gauge fields on Riemann surfaces [30].

3.2. TOPOLOGY OF THE ORBIT SPACE

The space of the gauge orbits $\mathcal{M}_\Sigma^G$ of connections defined on principal bundles over a Riemann surface Σ with structure group G is not connected in general. Each connected component contains the orbits of gauge fields defined on the same principal bundle $P(\Sigma, G)$. The set of connected components of $\mathcal{M}_\Sigma^G$ is therefore equal to the set of equivalent classes of principal bundles [30],

$$\pi_0(\mathcal{M}_\Sigma^G) = \check{H}^1(\Sigma, \mathfrak{G}). \tag{3.13}$$

This simplest way of describing $\check{H}^1(\Sigma, \mathfrak{G})$ is by means of homotopy classes $[\Sigma, BG]$ of maps from the Riemann surface Σ into the classifying space BG of G,

$$\pi_0(\mathcal{M}_\Sigma^G) = [\Sigma, BG].$$

Since BG is the base manifold of the universal bundle $EG(BG, G)$ [31] and EG is contractible, the following exact sequence of homotopy groups,

$$0 \equiv \pi_n(EG) \to \pi_n(BG) \leftrightarrow \pi_{n-1}(G) \to \pi_{n-1}(EG) \equiv 0, \tag{3.14}$$

establishes an isomorphism $\pi_n(BG) \equiv \pi_{n-1}(G)$ between the homotopy groups of BG and G. This yields the following results:

$$[\Sigma, B\mathrm{U}(1)] = \mathbb{Z}, \qquad [\Sigma, B\mathrm{SU}(N)] = 0.$$

In the case of abelian gauge fields the U(1) principal bundles are classified by the first Chern class $c_1(P)$. Higher homotopy groups of every component of $\mathcal{M}_\Sigma^G$ can be calculated in a similar way. The results are

$$\begin{aligned} \pi_1(\mathcal{M}_\Sigma^{\mathrm{U}(1)}) &= \mathbb{Z}^{2g}, \\ \pi_1(\mathcal{M}_\Sigma^{\mathrm{SU}(N)}) &= 0, \\ \pi_2(\mathcal{M}_\Sigma^{\mathrm{U}(1)}) &= 0, \\ \pi_2(\mathcal{M}_\Sigma^{\mathrm{SU}(N)}) &= \mathbb{Z}. \end{aligned}$$

The cohomology classes of $\mathcal{M}_\Sigma^G$ can be calculated in a similar manner by using Thom's theorem [30,33],

$$\begin{aligned} H^1(\mathcal{M}_\Sigma^{\mathrm{U}(1)}, \mathbb{Z}) &= \mathbb{Z}^{2g}, \\ H^1(\mathcal{M}_\Sigma^{\mathrm{SU}(N)}, \mathbb{Z}) &= 0, \\ H^2(\mathcal{M}_\Sigma^{\mathrm{U}(1)}, \mathbb{Z}) &= \mathbb{Z}^{g(2g-1)}, \\ H^2(\mathcal{M}_\Sigma^{\mathrm{SU}(N)}, \mathbb{Z}) &= \mathbb{Z}. \end{aligned}$$

In the abelian U(1) case the orbit space can be identified with [30]

$$\mathcal{M}_{\Sigma}^{U(1)} = \mathbb{Z} \times S^1 \times \overset{2g}{\cdots} \times S^1 \times P(H), \tag{3.15}$$

where $P(\mathcal{H}) = \mathcal{H}/U(1)$ denotes the projective space of rays of a separable Hilbert space $\mathcal{H}$. The $\mathbb{Z}$ factor describes the different connected components of $\mathcal{M}_{\Sigma}^{U(1)}$ characterized by the integer Chern number (magnetic charge) of the corresponding line bundles. The S^1 factors represent the moduli space of flat connections on Σ and $P(\mathcal{H})$ contains all transverse (non-flat) photon fluctuations. The first $\mathbb{Z}$ factor is responsible for the zero homotopy group $\pi_0(\mathcal{M}_{\Sigma}^{U(1)})$, the second T^{2g} factor for the first homotopy group $\pi_1(\mathcal{M}_{\Sigma}^{U(1)}) = \mathbb{Z}^{2g}$ and the last projective space $P(\mathcal{H})$ for the second homotopy group $\pi_2(\mathcal{M}_{\Sigma}^{U(1)}) = \mathbb{Z}$.

The above complete characterization of the abelian orbit space cannot be generalized to the non-abelian case. In this case, the generator of the second cohomology group $\check{H}^2(\mathcal{M}_{\Sigma}^{SU(N)}, \mathbb{Z})$ is the two-form [32,33]

$$\Omega(\tau, \eta) = -\frac{1}{16\pi^2}(\tilde{\tau}^h, *\tilde{\eta}^h) + \frac{1}{8\pi^2}(G_A * [\tilde{\tau}^h, \tilde{\eta}^h], *F(A)), \tag{3.16}$$

and the quantization condition of the Chern–Simons charge (3.11) can be easily understood from the identity

$$\Omega_\kappa = 2\pi k \Omega. \tag{3.17}$$

The topological structure of $\mathcal{M}_{\Sigma}^{SU(N)}$ is very similar to that of the configuration space of the quantum mechanical examples considered in section 2. In the genus zero case $\Sigma_0 = S^2$ the Yang–Mills theory is reminiscent of the magnetic monopole case. In that case we have shown the existence of an anomalous contribution to the angular momentum which transmutates the spin and statistics (bosonic/fermionic) of charged quantum particles moving around magnetic monopoles with odd magnetic charge. However, in the gauge theory case it has been shown that such a phenomenon does not occur [34]. Although the second homotopy group of $\mathcal{M}_{\Sigma}^{SU(N)}$ is non-trivial ($\mathbb{Z}$), the orbits of the SO(3) are contractible and do not enclose any magnetic charge. Therefore the only effect of the topological mass is to give an angular momentum $k/|k|$ to the massive gluon, which does not change its bosonic statistics [23].

3.3. VACUUM STRUCTURE

Another physical consequence of the non-trivial topology of the orbit space is the existence of nodes in all quantum states including the vacuum. The phenomenon arises because of the non-trivial character of the line bundle $\mathcal{E}_k(\mathcal{M}, \mathcal{C})$ where physical states are defined [34]. In pure 2 + 1 dimensional Yang–Mills theory Feynman argued that the absence of nodes in the vacuum functional might imply confinement [35]. He also argued that a generalization of the min–max principle would imply that the vacuum functional $\psi_0(A)$ does not vanish for any classical gauge field configuration. In the presence of the Chern–Simons term the min–max principle cannot be applied because of the complex character of the interaction as was pointed out in section 2 for finite dimensional quantum models. Moreover, in that case the deconfinement mechanism is based on the massive character of gluons, which is also due to the presence of the complex Chern–Simons term.

In a genus zero Riemann surface $\Sigma = S^2$ the locus of nodes of the vacuum functional can be obtained from the analysis of the gauge anomaly of chiral fermionic determinants. The effective gauge action of a chiral fermion $W(A) = \log\det(\not{D}_A)$ theory satisfies the gauge anomaly condition

$$d_A^* \frac{\delta}{\delta A} W(A) = \frac{\mathrm{i}}{8\pi} * dA, \tag{3.18}$$

which is identical to the Gauss law (3.6) constraint of physical states for $k = 1$ [32]. Therefore, any physical state of topologically massive Yang–Mills theory can be written as

$$\psi(A) = \mathrm{e}^{kW(A)} \xi(A), \tag{3.19}$$

$\xi(A)$ being an arbitrary gauge invariant normalizable functional. Thus, the locus of nodes of quantum vacuum states belongs to the set of the gauge field configurations A with vanishing fermionic determinant. When the spatial Riemann surface is a S^2 sphere these configurations correspond to gauge fields A whose associated complex structures are non-trivial, i.e., the hermitian connection A_z defined by $A_z = \frac{1}{2}(A_1 - \mathrm{i}A_2)$ is not a pure gauge $U^\dagger \partial U$ [36]. In such a case the set of gauge fields defined from global hermitian connections, $A_z = U^\dagger \partial U$, is dense in the space of gauge fields $\mathcal{A}$; thus, the existence of nodes at such configurations is only possible for the null state $\psi(A) = 0$.

The canonical theory is exactly solvable in the case of abelian gauge fields. The hamiltonian of Chern–Simons–Maxwell theory is quadratic in the gauge fields,

$$\mathbb{H} = -\tfrac{1}{2}\kappa^2 \Lambda \left\| \frac{\delta}{\delta A} + \frac{\mathrm{i}}{2\kappa^2} * A \right\|^2 + \frac{1}{2\kappa^2 \Lambda} \|dA\|^2, \tag{3.20}$$

and is very similar to the hamiltonian of the quantum Hall effect. The only differences come from the infinite dimensional character of the configuration space, the interaction terms $\|dA\|^2$ and the existence of the Gauss law constraint. However, since $\mathbb{H}$ commutes with the Gauss law operator

$$\mathbb{G} = d_A^* \frac{\delta}{\delta A} - \mathrm{i}\frac{1}{2\kappa^2} * dA$$

stationary physical states ($\mathbb{H}\psi = E\psi$, $\mathbb{G}\psi = 0$) can be constructed from linear combinations of eigenfunctionals of $\mathbb{H}$ with the same energy.

In the case of genus zero or the infinite plane $\mathbb{R}^2$ the orbit space is of the form $\mathcal{M} = \mathbb{Z} \times P(\mathcal{H})$, and if we restrict ourselves to the sector without magnetic charge, $\mathcal{A} = \mathcal{M} \times \mathcal{G}$ (there is no Gribov problem). Consequently, any gauge field $\boldsymbol{A}$ can be univocally split into its longitudinal and transverse components,

$$\boldsymbol{A} = \boldsymbol{A}^{\perp} + d\phi. \qquad (3.21)$$

Since in S^2 and $\mathbb{R}^2$ there are no harmonic one-forms, the transverse component of $\boldsymbol{A}$ is univocally defined by $\boldsymbol{A}^{\perp} = (I - d\Delta^{-1}d^*)\boldsymbol{A}$. Using the factorization (3.19) we can express the physical states in terms of gauge invariant functionals ξ,

$$\psi(\boldsymbol{A}) = e^{kW(\boldsymbol{A})}\xi(\boldsymbol{A}), \qquad (3.22)$$

with

$$W(\boldsymbol{A}) = -\frac{i}{8\pi}(*d\boldsymbol{A}, \Delta^{-1}d^*\boldsymbol{A}).$$

Using the splitting (3.21) and the factorization (3.22) the hamiltonian becomes [11]

$$\mathbb{H}' = -\tfrac{1}{2}\kappa^2\Lambda \left\| \frac{\delta}{\delta \boldsymbol{A}^{\perp}} \right\|^2 + \frac{1}{2\kappa^2\Lambda}\left(*d\boldsymbol{A}^{\perp}, (I + \Lambda^2/\Delta) * d\boldsymbol{A}^{\perp}\right) \qquad (3.23)$$

when acting on gauge invariant states ξ. The ground state is univocally given by [11]

$$\xi_0(\boldsymbol{A}) = \exp\left(-\frac{1}{2\kappa^2\Lambda}\left(*d\boldsymbol{A}, \Delta^{-1}(\Delta + \Lambda^2)^{1/2} * d\boldsymbol{A}\right)\right). \qquad (3.24)$$

The apparent infinite dimensional degeneracy associated to Landau levels is not present in this case because the Gauss law selects only one vacuum state (3.24). The ground state ξ_0 has infinite energy $E_0 = \mathrm{tr}(\Delta + \Lambda^2)^{1/2}$ as usual in quantum field theory. Once this vacuum energy is renormalized to zero, the topological limit ($\Lambda \to \infty$) leads to a one dimensional Hilbert space $\mathcal{H}_k^0 = \mathbb{C}$ with null hamiltonian $\mathbb{H} = 0$. The result agrees with the one obtained via canonical quantization of pure abelian Chern–Simons theory [10].

Higher energy states are given by the infinite dimensional generalization of Laguerre polynomials and correspond to the free propagation of an arbitrary number of spin one massive particles.

In the case of Riemann surfaces with higher genus it is necessary to introduce some changes in the above picture. The canonical quantization of pure Chern–Simons theory leads to a non-trivial quantum Hilbert space

$$\mathcal{H}_k^g = \mathbb{C}^{|k|^g}. \qquad (3.25)$$

The same result is obtained from the topological limit of the Chern–Simons–Maxwell theory. The degeneracy arises in this case because not all the degenerate Landau states are eliminated by Gauss law condition. In this case the transverse modes of the splitting (3.21) contain some harmonic forms $\boldsymbol{a}$ which generate the T^{2g} factor of the orbit space $\mathcal{M}_{\Sigma_g}^{U(1)}$. Therefore we have a more complex splitting of gauge fields degrees of freedom,

$$\boldsymbol{A} = \boldsymbol{A}'^{\perp} + \boldsymbol{a} + d\phi. \qquad (3.26)$$

The harmonic forms $\boldsymbol{a}$ must satisfy some periodic conditions to eliminate the overcounting of Gribov copies. The hamiltonian picks up an additional term governing the dynamics of these variables which is similar to the quantum Hall effect on T^{2g} with topological charge k. Consistency requires in this case that the Chern–Simons charge k be an integer [#3]. Therefore, ground states are given by the product of the functional (3.24) and the appropriate theta functions on T^{2g}. The vacuum degeneracy is $|k|^g$ in agreement with the pure Chern–Simons approach.

In the Chern–Simons–Maxwell theory the ground states do not depend on the holomorphic component $A_z = \frac{1}{2}(A_1 - \mathrm{i}A_2)$ of A. In fact, if we carry out the similarity transformation associated to holomorphic quantization,

$$\psi(A) = \mathrm{e}^{A_{\bar{z}}A_z/\kappa^2}\xi(A), \tag{3.27}$$

the hamiltonian becomes

$$\begin{aligned} \mathbb{H}' &= -\tfrac{1}{2}\kappa^2 \Lambda \left(\frac{\delta}{\delta A_z} - 2\kappa^{-2} * A_{\bar{z}} \right) \frac{\delta}{\delta A_{\bar{z}}} \\ &\quad + \frac{1}{2\kappa^2 \Lambda} ||dA||^2 + \Lambda \kappa^2 \,\mathrm{tr}\, I \end{aligned} \tag{3.28}$$

in the complex coordinates $A_z, A_{\bar{z}}$ of the configuration space $\mathcal{A}$ of gauge fields. The second term of $\mathbb{H}'$ does not leave invariant the space of holomorphic functionals. For this reason the ground states in the massive theory are not holomorphic. However, in the topological limit this term vanishes and we recover the holomorphic structure of pure Chern–Simons quantum states.

The quantization in non-trivial magnetic backgrounds $c_1(P) = n$ is also consistent when the Chern–Simons charge k is an integer [38]. The degeneracy of the ground state is metric independent and in the case of even topological charges k modular transformations act trivially on the subspace of ground states over the T^2 torus .

The infinite volume case $\mathbb{R}^2$ is similar to the genus zero S^2 case and the ground state is unique, as we have shown above. However, it can also be obtained from that of any Riemann surface when its volume increases to infinity. For any finite volume the associated torus T^{2g} of the orbit space has a constant volume $(2\pi|k|)^g$ which counts the degeneracy of the quantum vacuum states. However this torus shrinks to a single point in the infinite volume case and in that case the quantum vacuum state becomes unique. The other quantum states tend to

[#3] The Gauss law only imposes invariance under infinitesimal gauge transformations. If we do not impose invariance under large gauge transformations it is possible to obtain a consistent quantization for any value of k. This is equivalent to considering all the Gribov copies of the fields $\boldsymbol{a}$ as inequivalent field configurations. In such a case the relevant orbit space is $H^1(\Sigma_g, \mathbb{R}) \times P(\mathcal{H})$ and the space $\mathcal{H}^g_k$ becomes infinite dimensional. When k is a rational number $k \in \mathbb{Q}$ the symmetry group contains some large gauge transformations and the Hilbert space remains finite dimensional but with higher degeneracy than $|k|^g$ [37].

the same state and the degeneracy disappears in that limit. In the case $g = 1$ this limit corresponds to the quantum Hall effect when the size of the torus shrinks to a single point keeping the magnetic flux constant. The geometric interpretation is that in the present case the Gribov horizon becomes closer and closer to the origin along the harmonic forms directions, because the gauge transformations

$$\phi(x) = e^{ix\cdot\boldsymbol{a}} \tag{3.29}$$

shift the gauge variable by a constant harmonic form,

$$A^{\phi} = A + \boldsymbol{a}, \tag{3.30}$$

and the periodic boundary conditions require $\boldsymbol{a}$ to be on the dual torus. In the infinite volume limit that torus T^2 becomes a single point because any gauge transformation of the type (3.29) belongs to $\mathcal{G}_{\mathbb{R}^2}$ for any value of $\boldsymbol{a}$. The only apparent reminiscence of the degeneracy is the existence of long range correlations in the two point functions of massive photons [26]. But such correlations are rather associated to some (infinitly massive) null states. In some sense this phenomenon is opposite to the Goldstone mechanism; there are no local physical excitations associated to the long range correlation functions.

In the non-abelian case the pure Chern–Simons theory is also exactly solvable for compact Lie groups G. The Hilbert space of quantum states for $G = SU(N)$ is [40]

$$\mathcal{H}^0_{S^2} = \mathbb{C}, \qquad \mathcal{H}^0_{T^2} = \mathbb{C}^{n(N,k)},$$

where $n(N,k) = \binom{N+k-1}{k}$ is the number of weights in a fundamental k-alcove $\Lambda^w/k\Lambda^r \ltimes W$ of $SU(N)$. In the infinite volume limit the quantum Hilbert space also becomes one dimensional as in the abelian case.

In topologically massive Yang–Mills theory the spectrum of the hamiltonian is not exactly known, but we can get some information from the pure topological phase. In the infinite volume limit the vacuum state is unique because the corresponding topological limit has no degeneracy. The same property holds when the spatial manifold is a two dimensional sphere. However, in the case of a two dimensional torus the degeneracy of the vacuum is unknown. It is certainly lower than $n(N,k)$ because of the potential term

$$\frac{1}{2\kappa^2\Lambda}||F(A)||^2,$$

which in this case breaks the degeneracy of Chern–Simons states. Since this term becomes irrelevant in the topological limit, we recover the Chern–Simons degeneracy when $\Lambda \to \infty$. This illustrates how the existence of a solvable topological phase can shed some light in the vacuum structure of a full fledged quantum field theory.

It is also interesting to analyze how the dynamic constraints of the Chern–Simons theory appear in the topological phase of a massive gauge theory. The only constraint in the massive theory is the Gauss law (3.6). However, in the topological limit the ground states tend to holomorphic functionals $\xi(A_z)$ and the operators $A_{\bar{z}}$ and $-2\kappa^{-2}\,\delta/\delta A_z$ can be identified in a weak sense. Their expectation values on holomorphic states are equal. In this way the second class momentum constraints arise in the topological limit in a similar manner to the quantum mechanical models of section 2. However, in this case we have an additional constraint $\mathbb{G}\xi(A_z) = 0$, which with the above identifications becomes

$$(\xi_1(A_z), \mathbb{G}\,\xi_2(A_z)) \equiv (\xi_1(A_z), *F(A_z)\,\xi_2(A_z)) = 0 \qquad (3.31)$$

in a weak sense. Therefore ordinary gauge invariance in the topological phase becomes holomorphic gauge invariance, which is the higher symmetry that makes Chern–Simons theory solvable.

4. Covariant functional integral approach

In previous sections our analysis of topological quantum theories was based on a canonical operator formalism. Most of the results can also be derived by path integral methods. In particular, it is very interesting to see how the main properties of the topological quantum mechanical systems analyzed in section 2 can be obtained by means of the path integral approach, which, on the other hand, illustrates the low energy features described in the introduction. However, the description of the path integral quantization would enlarge excessively the content of this note and will be given elsewhere.

In the case of Chern–Simons theory the covariant functional integral method exhibits some new features which merit a separate discussion. Although the theory is finite and solvable in the canonical formalism for compact Lie groups in the absence of space–time boundaries and external sources, in the covariant formalism it is simply renormalizable.

The existence of singularities is due to the presence of some unphysical degrees of freedom in this formalism. Therefore, the analysis becomes non-trivial and to some extent the fluctuations of those unphysical degrees of freedom might veil the topological nature of the theory.

On the other hand, Witten conjectured that some quantum observables of Chern–Simons theory provide field theoretical definitions of topological invariants of knots, links and three-manifolds [39]. This beautiful idea has been pushed forward by different methods in the canonical formalism [40,21]. However, the analysis of the conjectures for a general three dimensional manifold requires the use of a covariant formalism. Moreover, the analysis of possible gravitational or framing anomalies is not complete in the canonical formalism because of the special form of the space metric in such a formalism: direct product of a two dimensional riemannian metric and an one dimensional time scale. For instance, the induced gravitational Chern–Simons term is not metric dependent for such metrics, and vanishes for some choices of framing. Although the analysis of those anomalies can be achieved in an indirect way by using topological techniques like topological surgery, instead of field theoretical techniques, a complete discussion in terms of pure field theoretical arguments requires the use of a three dimensional covariant approach.

The first problem in the covariant formalism is the existence of ultraviolet divergences. Therefore, it is necessary to introduce some regularization in order to smooth the ultraviolet behavior of the Chern–Simons interaction. We shall use a geometrical regularization which preserves many of the interesting properties of the model: continuity of space–time and invariance under framing and gauge transformations.

4.1. GEOMETRIC REGULARIZATION

The essential characteristics of geometric regularization are based on the observation that, because of gauge invariance, the relevant space of covariant field configurations is also the space of covariant gauge orbits $\mathcal{M}$. Since $\mathcal{M}$ is a curved infinite dimensional riemannian manifold the regularization of a functional integral defined over $\mathcal{M}$ does not simply require a regularization of the action, as in ordinary field theories with flat configuration spaces, but also a non-trivial regularization of the functional (riemannian) volume element. In this way it is possible to obtain a regularization which preserves the topological properties of continuum approaches and has a non-perturbative interpretation.

Because of the pseudoscalar character of the Chern–Simons action, standard perturbative regularization methods cannot be applied. This fact has recently stimulated the interest in the application of different perturbative regularization prescriptions to Chern–Simons theories [27,41–43].

The geometric regularization method proceeds by three steps [42,44].

(1) *Regularization of the classical action* by means of a Yang–Mills term with higher covariant derivatives:

$$S_\Lambda(A) = \frac{ik}{4\pi}\int_{M_3} \operatorname{tr}(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A) + \frac{k}{8\pi\Lambda}(F(A), (I + \Delta_A/\Lambda^2)^n F(A)). \tag{4.1}$$

(2) *Regularization of the volume of gauge orbits*, $\det^{1/2}\Delta_A^0 = \det^{1/2} d_A^* d_A$, by the Pauli–Villars method,

$$\det{}^{1/2}_{\Lambda\Lambda'}\Delta_A^0 = \det{}^{-1/2}(I + \Delta_A^0/\Lambda'^2)^{2n+2} \times \det{}^{1/2}\Delta_A^0(I + (I + \Delta_A^0/\Lambda^2)^{2n})\Delta_A^0/\Lambda^2). \tag{4.2}$$

And the crucial step

(3) *Regularization of the volume element of the covariant gauge orbit space* $\mathcal{M} = \mathcal{A}/\mathcal{G}$ in terms of a binuclear riemannian structure $(g^0_{A'}, G^1_{A'}, G^2_{A'})$ of $\mathcal{M}$. It consists of a riemannian metric $g^0_{A'}$ and two families of self-adjoint trace class operators $G^1_{A'}, G^2_A$ acting on the tangent spaces of $\mathcal{M}$. The nuclear structure is the basic functional structure behind the construction of Gaussian measures in Hilbert spaces. The generalization of this structure turns out to be very relevant for the construction of functional measures in Hilbert Lie group [45] and arbitrary infinite dimensional Hilbert manifolds including gauge orbit spaces [46]. Geometric regularization is based on the application of this construction to the covariant formalism [28]. In this case the binuclear riemannian structure can be defined by means of the three riemannian metrics $g^i_{A'}, i = 0, 1, 2$, of $\mathcal{M}$ given by

$$g^i_{A'}(\tau, \eta) = (P_A\tilde{\tau}, (I + \Delta_A/\Lambda'^2)^{m_i} P_A\tilde{\eta}), \quad i = 0, 1, 2, \tag{4.3}$$

for any tangent vectors $\tilde{\tau}, \tilde{\eta}$ of $T_A\mathcal{A}$ whose projections on $\mathcal{M}$ are τ and η, respectively. Then, the operators $G^i_{A'} : T_{[A]}\mathcal{M} \longrightarrow T_{[A]}\mathcal{M}$ $(i = 1, 2)$ defined by

$$g^i_{A'}(\tau, \eta) = g^0_{A'}(\tau, (G^i_{A'})^{-1}\eta) \tag{4.4}$$

are self-adjoint and trace class with respect to $g^0_{A'}$ for $m_i \geq m_0 + 2$, and define a binuclear riemannian structure in the orbit space $\mathcal{M}$.

The geometric regularization of the functional integral is given by

$$\int_{\mathcal{M}} \delta\mu_{g^0_{A'}, G^1_{A'}, G^2_{A'}}([A]) \, \det{}^{1/2}_{\Lambda\Lambda'} \Delta^0_A \, e^{-S_\Lambda(A)}, \tag{4.5}$$

where $\delta\mu_{g^0_{A'}, G^1_{A'}, G^2_{A'}}([A])$ is the regularized functional volume element associated to the binuclear structure $(g^0_{A'}, G^1_{A'}, G^2_{A'})$, which, e.g., in the generalized Landau gauge

$$d^*_{A_0}(A_c - A_0) = 0 \tag{4.6}$$

reads

$$\delta\mu_{g^0_{A'}, G^1_{A'}, G^2_{A'}}(A_c) = \delta A_c \det{}_c^{1/2} g^0_{\Lambda\Lambda'}(A_c) \times \det{}_c^{1/2}(G^1_{A'})^{-1} \det{}_c^{1/2}(G^2_{A'})^{-1} G^1_{A'}. \tag{4.7}$$

In the limit $(\Lambda, \Lambda') \rightarrow (\infty, \infty)$ we recover the Babelon–Viallet expression [47] for the functional integral,

$$\int \delta A_c \det{}^{1/2} g(A_c) \det{}^{1/2} \Delta^0_A e^{-S(A)}, \tag{4.8}$$

which is equivalent to the standard one obtained by means of the Faddeev–Popov mechanism, because

$$\det{}^{1/2} g(A_c) \det{}^{1/2} \Delta^0_A = \det d^*_{A_0} d_A.$$

In this limit the binuclear structure disappears, $(g^0_{A'}, G^1_{A'}, G^2_{A'}) \rightarrow (g, I, I)$, and $\mathcal{M}$ becomes a weak riemannian manifold with metric g.

4.2. FINITENESS

There are some further restrictions on the exponents of the regulators which are necessary for consistency. In flat space–times the regularized Green functions are finite if and only if

$$n + 1 = m_2, \qquad n > \{1, m_0, m_1 - m_0, m_1 - m_2\}. \tag{4.9}$$

Since those finite results are obtained by cancellation of one-loop divergences a complete formulation of the regularization with non-perturbative interpretation requires the introduction of a non-perturbative auxiliar (gauge dependent) cut-off. Therefore, in order to recover gauge invariance after the removal of the auxiliar cut-off the exponents of the regulators must satisfy the following condition [44]:

$$(n + 1)^2 - m_0^2 - (m_1 - m_0)^2 - (m_2 - m_1)^2 = 0, \tag{4.10}$$

which guarantees that Slavnov–Taylor identities are satisfied in perturbation theory. When the conditions (4.9), (4.10) are satisfied the functional integral is completely regularized in a global gauge invariant way. There is an infinite family of integer solutions of the constraint equations (4.9), (4.10) which give rise to consistent geometric regularizations.

One loop calculation of the Chern–Simons effective action with geometric regularization yields [42]

$$\Gamma^{(1)}(A) = \Gamma^{(1)}_{\rm nl}(A) + \mathrm{i} \int \mathrm{tr}(\alpha_2 A \wedge dA + \tfrac{2}{3}\alpha_3\, A \wedge A \wedge A), \tag{4.11}$$

with

$$\alpha_2 = (\tfrac{1}{2}\pi + \tfrac{4}{3}I_n)\frac{N}{2\pi^2}, \qquad \alpha_3 = (\tfrac{1}{2}\pi + 2I_n)\frac{N}{2\pi^2},$$

$$I_n = \int_0^\infty \mathrm{d}p\, \frac{(1 + p^2)^n}{1 + p^2(1 + p^2)^{2n}},$$

and $\Gamma^{(1)}_{\rm nl}(A)$ being a non-local scalar term associated to a global anomaly.

The fact that $\alpha_2 \neq \alpha_3$ implies the existence of a finite renormalization of the gauge field which is not universal [42,44] [$A_{\rm R} = Z^{1/2}A$, with $Z = 1 + 2(\alpha_3 - \alpha_2) + \mathrm{O}(1/k)$]. The Chern–Simons charge k is also renormalized by a finite universal additive constant $k_{\rm R} = k + N$. Both renormalizations of the gauge field and the coupling constant are in agreement with the one loop renormalization of gauge transformations and Slavnov–Taylor identities [42,44]. Although we know by general symmetry arguments that the theory is finite to any order in perturbation theory [48], the explicit renormalizations obtained above are characteristic of all regularizations involving a Yang–Mills term which breaks the pure pseudoscalar character of the action. In particular, the renormalization of the Chern–Simons coupling constant makes possible the identification of Witten's charge with the bare charge in the definition of topological invariants. The calculation of the Wilson loop expectation value for an unknotted loop up to sixth order in perturbation theory [50] shows the agreement of the above prescription with Witten's results [#4]. A two loop calculation in the case $n = 0$ with an auxiliar dimensional regularization recently carried out also shows that there are no additional corrections to the value of the renormalized charge [52], which presumably indicates that the same result holds to any order of perturbation theory in geometric regularization.

4.3. METRIC DEPENDENCE

Let us now analyze the metric dependence of the quantum Chern–Simons theory in the framework of geometric regularization. Although the classical lagrangian is metric independent, a metric dependence might appear in the quantization process from the functional measure δA_c [53] or the gauge fixing condition because both are metric dependent. Moreover, geometric regularization is defined by a binuclear riemannian structure of the orbit space which is based on the space–time metric and this dependence could remain even after the removal of the ultraviolet regulators $(\Lambda, \Lambda' \to \infty)$.

In order to analyze this dependence we look, as in sections 2 and 3, at the most sensitive observable under changes of space–time metrics: the partition functional.

The one loop contribution given by

$$\begin{aligned} \mathcal{Z}^{(1)}(g) = {} & \det{}_c^{1/2}(I + \Delta^1/\Lambda'^2)^{n+1} \\ & \times \det{}^{1/2}\left(\Delta^0/\Lambda^2(I + \Delta^0/\Lambda^2)^{2n} + 1\right)\det{}^{1/2}\Delta^0 \\ & \times \det{}_c^{-1/2}\left[\Delta^1/\Lambda^2(I + \Delta^1/\Lambda^2)^n + \mathrm{i} * d\right] \\ & \times \det{}^{-1/2}(I + \Delta^0/\Lambda'^2)^{2n+2} \end{aligned} \tag{4.12}$$

in the generalized Landau gauge is finite and can be evaluated in the weak limit approximation $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. The result is of the form [54]

$$\begin{aligned} \mathcal{Z}^{(1)} = {} & \mathcal{Z}_0^{(1)}(g)\exp\left(c_1\int_{M_3}\sqrt{g} + c_2\int_{M_3}\sqrt{g}R\right) \\ & \times \exp\left(\frac{\mathrm{i}c_3}{4\pi}\int_{M_3}\mathrm{Tr}(\omega\wedge\omega + \tfrac{2}{3}\omega\wedge\omega\wedge\omega) + \mathrm{O}(1/\Lambda)\right), \end{aligned} \tag{4.13}$$

[#4] Those results are obtained using the renormalized correlation functions of the Chern–Simons theory (4.11). A similar calculation in the framework of geometric regularization leads to slightly different metric dependent results [51].

where $\mathcal{Z}_0^{(1)}(g)$ contains the non-local terms induced by the global framing anomaly [39] and non-perturbative contributions generated by the existence of zero modes [54]. We remark that once the Green functions are finite in flat space–times they remain finite for arbitrary space–time metrics. Therefore there are no further restrictions on the parameters of geometric regularization.

The only terms which remain finite in the ultraviolet limit, $\Lambda, \Lambda' \to \infty$, are the gravitational Chern–Simons term c_3 and $\mathcal{Z}_0^{(1)}(g)$. The other terms would require counterterms to cancel their divergent contributions, $c_1 = \mathrm{O}(\Lambda^3) + \mathrm{O}(\Lambda'^3)$ and $c_2 = \mathrm{O}(\Lambda) + \mathrm{O}(\Lambda')$.

The explicit calculation of the coefficients of the induced gravitational action (4.13) yields the following values:

$$c_1 = 0, \quad c_2 = \tfrac{1}{16}(N^2-1)(\alpha\Lambda - \pi(n+1)\Lambda'), \quad c_3 = \tfrac{1}{24}(N^2-1), \tag{4.14}$$

for SU(N) in the generalized Landau gauge (4.6), with

$$\alpha = \int_0^\infty \mathrm{d}p \, \frac{1 + 2n(1 + p^2(1+p^2)^{2n-1})}{1 + p^2(1+p^2)^{2n}}. \tag{4.15}$$

The coefficient c_2 of the Einstein–Hilbert term depends on the parameters of the regularization and can be cancelled by a suitable choice of the regulator masses, $\Lambda' = \alpha\Lambda/\pi(n+1)$. However, the coefficient of the Chern–Simons term is universal and cannot be cancelled by any choice of the parameters of the regularization.

The value of $c_3 = (N^2-1)/24$ is in agreement with the exact value conjectured by Witten,

$$c_3 = \frac{k(N^2-1)}{24(k+N)} = \frac{N^2-1}{24}(1 - N/k + \mathrm{O}(N^2/k^2)). \tag{4.16}$$

A two loop calculation of c_3 has been recently carried out in a different perturbative regularization scheme [49]. The result gives the second term of Witten's expansion (4.16), but in such a scheme the first term is missing.

In the geometric regularization scheme there is no framing anomaly because the change of the gravitational Chern–Simons term under non-trivial framing transformations is compensated by the change of the non-local part of $\mathcal{Z}_0^{(1)}(g)$. In fact, geometric regularization preserves framing independence explicitly. But, consequently, the partition function becomes metric dependent. The cancellation of this dependence can only be achieved by the addition of a finite gravitational Chern–Simons counterterm which induces a framing anomalous behavior of the quantum partition function. In this sense metric dependence can be traded by a framing anomaly.

The one loop contributions to the partition function $\mathcal{Z}^{(1)}(g)$ can also be exactly calculated beyond the weak field expansion for some particular space–time backgrounds ($S^3, \Sigma \times S^1$) [54]. The results confirm the values of the coefficients

(4.16) of the local terms of the induced action: cosmological, Einstein–Hilbert and gravitational Chern–Simons terms, and provide explicit expressions for the non-local terms of $\mathcal{Z}_0^{(1)}(g)$. In the case of a manifold of the form $\Sigma \times S^1$ the metric dependence of those non-local terms cancels out with that of the non-perturbative contributions of $\mathcal{Z}_0^{(1)}(g)$, up to a constant factor $(\mathrm{vol}\, M_3)^{1/2}$ [55], which is genus independent [54]. This fact stresses the topological character (metric independent) of Chern–Simons theory in the canonical formalism. Because of the direct product structure of the space–time metric in this formalism the gravitational Chern–Simons term is not metric dependent and there always exists a framing where it vanishes. Once the metric dependent factor is eliminated the partition function of the abelian theory $\mathcal{Z}^{(1)}(g) \sim k^{g+1}$ gives account of the vacuum degeneracy k^g [54]. In the non-abelian case it gives only the leading approximation to the exact vacuum degeneracy [54]. It will be very interesting to know whether higher order perturbative calculations agree with higher order corrections to the exact formula.

Although there is not any apparent symmetry argument behind the choice $\Lambda' = \alpha\Lambda/\pi(n+1)$ of the masses of the regulators, it is a necessary condition to cancel an explicit metric dependence which arises from the quantum fluctuations. The partition function is not the only observable which picks up quantum metric dependent contributions [51]. In general, it is always possible to factorize the metric dependent contributions and obtain a topological invariant, but this procedure has to be carried out very carefully because some relevant topological information can also be lost by a crude renormalization. This necessity of renormalization introduces some ambiguities in the definition of topological invariants in Chern–Simons theory.

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